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CRT-2002-16

Centre de recherche sur les transports (C.R.T.)
Centre for Research on Transportation

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June 2002

PUBLICATION
CRT-2002-16

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* This research was supported in part by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), an agency of the Brazilian Federal Government, and by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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Dépôt légal – Bibliothèque nationale du Québec
Bibliothèque nationale du Canada, 2002

103-C3063/

ABSTRACT

The production of petroleum refined products in refineries results in the output of a main product and several byproducts. Conventional oligopoly models cannot be applied to analyze oligopolies in the petroleum industry because they do not consider interaction among products on the supply side. A spatial oligopoly equilibrium model is proposed to analyze the Brazilian market, where the number of companies supplying those products is small. The profit maximizing functions of each firm are formulated by considering the optimization of the operations of refineries, as well as the crude oil production of each company. A given refinery may use crude oil, in the refining process, from the production facilities of the company, or buy other types of crude oil on the world market. An algorithm based on the Gauss/Seidel Method and on a relaxation algorithm for solving variational inequalities is developed to find the equilibrium prices and trading flows of the refined products. At each iteration, a profit maximizing nonlinear problem is solved for each firm, by considering that the strategies (production levels) of the other firms are fixed. It is assumed that each firm uses the last information available. The Brazilian market, where the state monopoly is being opened to competition this year, is analyzed. Policy interventions on prices are also analyzed.

Key words: oligopoly equilibrium, nonlinear programming, petroleum refineries.

RÉSUMÉ

La production des produits de raffinage du pétrole dans les raffineries a comme conséquence le rendement d'un produit principal et de plusieurs sous-produits. Des modèles conventionnels d'oligopole ne peuvent pas être appliqués pour analyser des oligopoles dans l'industrie du pétrole parce qu'ils ne considèrent pas l'interaction parmi des produits du côté de l'offre. On propose un modèle spatial d'équilibre d'oligopole pour analyser le marché brésilien, où le nombre de compagnies fournissant ces produits est petit. Les fonctions de maximisation de profit pour chaque entreprise sont formulées en considérant l'optimisation des opérations des raffineries, ainsi que la production de pétrole brut de chaque compagnie. Une raffinerie donnée peut employer du pétrole brut provenant des opérations de la compagnie même, ou peut acheter d'autres types de pétrole brut sur le marché mondial. Un algorithme inspiré par la méthode de Gauss/Seidel en utilisant une approche de relaxation pour résoudre des inégalités variationnelles est développé pour trouver les prix d'équilibre ainsi que les flots commerciaux d'équilibre des produits de raffinage. À chaque itération, un problème nonlinéaire de maximisation de profit est résolu pour chaque entreprise, en considérant que les stratégies (niveaux de production) des autres sociétés sont fixes. On suppose que chaque entreprise emploie la dernière information disponible. Le marché brésilien, où le monopole d'état est ouvert à la concurrence cette année, est analysé. L'intervention de politiques sur les prix est également analysée.

Mots clés: équilibre d'oligopole, programmation nonlinéaire, raffineries de pétrole.

1. Introduction

Over the past 50 years, the petroleum industry in Brazil was managed as a state monopoly. The government created a company called PETROBRAS that was awarded all the rights for importing and producing petroleum products in Brazil. During this period of time, PETROBRAS has expanded and increased its crude oil production as well as its refining capacity. At the present time, PETROBRAS can supply with its own refineries 90% of the Brazilian demand for petroleum refined products.

With the object of promoting the development of regions remote from the location of the refineries, which are located in the more developed regions of the country, the consumer prices of refined products have been set uniformly by the government without differences among the various regions. This policy was supported by a cross subsidy mechanism. The more developed regions, which have lower supply costs, have provided the subsidy to the distant regions, which have a higher supply cost.

During the last decade, a great part of these subsidies have been eliminated. However, in some remote and undeveloped regions, especially some areas of the Amazon Rain Forest, this subsidy mechanism is still in effect. Without this subsidy, the price would be very high and the consumers would use other fuels, which may be obtained locally, such as wood from the forest. This would cause a deceleration of the development of such regions and may increase the forest degradation.

Currently, the government is opening the petroleum products market. The only company licensed to import refined products was PETROBRAS. The government has established the Brazilian ex-refinery prices and they were not dependent on an alternative external supplier. However, this situation has changed and other companies may import those products and deliver them directly to the consuming regions. The government no longer establishes the ex-refinery prices.

Considering the new market environment, there is concern about how the new prices for refined products can be set in the country. This concern is even greater when analyzing the probable need of subsidies to remote regions, even in the case that these regions would be supplied from foreign origins.

One tool that could be used to estimate these new prices is the Spatial Price Equilibrium Model (Samuelson, 1952). However, considering the special characteristics of the petroleum industry and the fact that the market in Brazil is not totally competitive, this model would be inadequate to represent this reality. Therefore, a multi-product spatial price oligopoly equilibrium model is probably more adequate. This multi-product model must consider the special characteristics of the production of refined products in refineries. The production of each refined product is related to the production of the others, depending on the crude oil, operating mode and complexity of the refinery.

In the next section, a brief review of the oligopoly models found in literature is presented. In section 3, the formulation of the profit maximization problem for petroleum companies is

presented. Then, an algorithm to find oligopoly equilibrium among firms with such profit functions is proposed in section 4. Section 5 presents the Brazilian petroleum market and some results of the equilibration algorithm used, and finally the conclusions and possible extensions are presented in section 6.

2. Oligopoly Market Equilibrium – pertinent literature

One of the main assumptions of the spatial price equilibrium model is that the supply of commodities occurs in totally competitive markets, where no firm unilaterally can impact the prices with a change in its production strategy. Hence, demand price equals supply price in every region and the markets clear at equilibrium[1]. However, this notion of perfectly competitive markets is often unrealistic in many instances, such as the markets for oil and coal. It also ignores the existence of local spatial monopolies, where a firm has complete local control of the demand of a small region.

A more suitable paradigm for such markets is that of spatial oligopoly market equilibrium. In the oligopoly market equilibrium, there are F firms that seek to maximize their own individual profits. Each firm has perfect knowledge of the demand and tries to maximize its own profit considering the strategies (production level) of the other firms.

The classical oligopoly model considers a demand market and F firms supplying a homogenous product in a noncooperative fashion. A Nash equilibrium is established when all the F firms set output levels in such a way that no firm alone can increase its own profit, producing more or less products. This problem is usually formulated as a variational inequality, considering the marginal profit of each firm. More detailed definition of this model can be found in Murphy *et al* (1982), Harker (1984) and Nagurney (1993).

Harker (1986) and Nagurney (1993) present a spatial oligopoly model where the profit function of each firm considers the transportation costs from the production plants to the demand markets. Harker (1986) compares the spatial oligopoly game with the spatial price equilibrium, totally competitive market, and with two possible monopoly situations. All the four approaches have a network structure, and can make use of network algorithms to find the equilibrium. As expected, in the example presented, the oligopoly game profits, prices and flows are between those of the spatial price equilibrium and those of the monopoly model solution.

Okuguchi & Szidarovsky (1999) present a local oligopoly game with multi-product firms. Each firm has a cost function depending on the manufacturing of all products. In a general approach, a given limited input can be used to produce different products, then, the production levels of the products can depend on each other. However, when formulating the problem as a nonlinear complementarity problem, it is assumed that the production levels do not depend on each other and, for each product, they are in the interval $[0, L_p]$, where L_p is the firm's production capacity for product p. Then, the set of all feasible output vectors X_f, for each firm f, equals the Cartesian product of the one-dimensional intervals $[0, L_p]$.

[1] Given the equilibrium prices, the produced quantity is equal to the demanded quantity.

All these contributions consider that the profit functions are continuously differentiable. Flam & Ben-Israel (1990), in the classical oligopoly problem, assume that the profit function need only be supper-differentiable, i.e., piecewise concave. It allows, for example, the use of profit functions obtained from mathematical programming, where, for example, different production plants with different efficiencies are used. Functions like these are not continuously differentiable, and the marginal profit is a step function, starting with the plants with better performance, using them until capacity is achieved, then using other plants with lower marginal profit and so on.

Equilibration algorithms for oligopoly models usually make use of diagonalization/relaxation algorithms used to solve variational inequalities. These algorithms applied to oligopoly problems have an interesting interpretation as a Cournot expectations dynamic process, (Harker, 1984 and Okuguchi & Szidarovsky, 1999). This process assumes that each player (firm) establish its strategy (production level) by trying to maximize its individual profit, considering that the other players will set their own strategies as the same as before. After all the firms make their choice, the game moves to another state and the firms play again, until no firm can improve its profit.

Okuguchi & Szidarovsky (1999) present other dynamic processes for oligopoly equilibration. The sequential adjustment process assumes that each player uses the last information available, instead of the information obtained in the previous time period. This permits a firm to use the information from the production level of another firm, which has already played in the current time period. This process, which can be interpreted as a Gauss-Seidel method, has usually faster convergence than the Cournot expectation process, which is interpreted as a Jacoby method. Consider that q^t_f is the strategy of firm $f=1,..,F$, at iteration t. In summary, each firm in the Jacoby method has a profit function $U^t_f(q^{t-1}_1,\ldots,q^{t-1}_{f-1},q_f,q^{t-1}_{f+1},\ldots,q^{t-1}_F)$ to be maximized considering the strategies of other firms obtained in the previous iteration, while that in the Gauss-Seidel method, the profit function is defined by $U^t_f(q^t_1,\ldots,q^t_{f-1},q_f,q^{t-1}_{f+1},\ldots,q^{t-1}_F)$, and considers the strategies of the firms that have already played in the current iteration.

3. Spatial Oligopoly Game with Petroleum Companies

There are different types of petroleum companies. In this paper, the market analyzed is the refined products market. Firms that only produce crude oil from petroleum basins are not considered here. Rather, firms that operate refineries producing refined products to sell to market are the main focus of this study. Other types of firms that do not operate refineries, but produce some refined products in other kind of industrial plants, such as petrochemical plants, or blend and trade those products are also included in the analysis, since they are potential competitors to the refiners.

When considering the formulation of the profit maximization problem of each firm, the more complex firms are the ones that operate several refineries, spatially separated, and also produce crude oil. Such a firm can refine its own crude oil in its refineries, sell the crude oil on the global market and buy other kinds of crude oils to refine in its refineries. The formulation of the profit maximization problem for other types of firms can be considered as special cases of this one.

The petroleum refined products oligopoly problem is then defined as the game played by these firms, where each firm tries to maximize its own profit considering the strategies (production levels) of the other firms for each product and demand market. This section discusses the profit maximization problems of refining companies and their special characteristics.

3.1 Profit Maximization Problem for Refining Companies as a Linear Programming

For simplicity, a profit maximization problem of a petroleum company is formulated first as a linear programming problem and, in this case, the price and the demand for each product at each market are considered to be fixed. Because of this fixed demand, an explicit maximum demand constraint is defined for each demand sub-region. For the external demand sub-region (refined products can be exported), the price functions are assumed to be constant, then, the maximum demand constraint is relaxed for this sub-region. This is not an unrealistic assumption since the amount exported must be small compared to the total demand in the global market.

The assumptions made in order to formulate the model are the following. Each refinery has its own characteristics, producing different product mixes from the same crude oil, with different operational costs. Each one has a local market, where refined products are sold at "better" prices. Finished refined products can be swapped among the refineries, and a pipeline network sometimes links the refineries. The transportation costs among refineries that make use of the pipeline network are usually low. The local demand of each refinery can be met with the local production or with products from other refineries (or imported products – from other firms).

Crude oil types are sub-divided into two sub-groups: $C1$ contains the crude oils produced at the production facilities of the company (each production facility produces only one type of crude oil); and $C2$ contains the other crude oils available on the market. The acquisition costs of crude oils in the sub-group $C1$ are the production cost cp_i (at the facility i) plus the transportation costs to each refinery ct_{ij} (from production facility i to refinery j). Each production facility i has a production capacity P_i. For crude oils in the sub-group $C2$, the acquisition costs are the CIF prices of each one at a virtual point on the boundary of the region in analysis, plus transportation costs to each refinery: $pr_i + ct_{0j}$. Constraints on the supply capacity of crude oils in $C2$ can also be modeled, assuming that their external prices are fixed. Crude oils from $C1$ can be sold on the global market. The same virtual point on the boundary is used to evaluate FOB prices pr_i for crude oils in $C1$ when they are exported (or sold to another company in the analyzed region). Eventual transportation costs in these transactions must be considered in the profit maximization problem (for example, if a production facility i is far from the virtual exporting point, incurring an extra transportation cost).

Each refinery has a given capacity, R_j, to refine crude oil. The production of some amount of one "profitable" product, such as gasoline, implies the production of a residual product, such as fuel oil. The quantity of each one that will be produced depends on the characteristics of the refinery, on the crude oil and on the operating mode of the refinery, which in turn depends on the quality of the crude oil. This is formulated by using constraints on the conversion rates.

The solution of this problem yields the optimal production plan for the refineries and crude oil production facilities. The time horizon considered is short term; in other words, no increase in

production or refining capacity is considered, but only the capacities already installed. Therefore, the crude oil production costs and the refining costs must consider only the variable costs. This means that return on investments are not considered, because they will not affect the profitability of the firm in the short term, once the capital has already been spent. Thus, only the costs really involved in the day-by-day production activities are considered.

The notation used in this model formulation is now presented. n_j represents the number of refineries. n_k is the number of demand sub-regions (including export). n_p is the number of refined products (including residual products). $n_{om(i)}$ is the number of operating modes for each crude oil and refinery. $z^{p,j}_{i,om(i)}$ is the quantity of product p produced at refinery j, from crude oil i, at operating mode $om(i)$. $y^{p,j,k}$ is the quantity of product p produced in refinery j, and sold at the demand sub-region k. The quantity of crude oil i refined at refinery j, at operating mode $om(i)$, is represented by $x^j_{i,om(i)}$. x^e_i is the quantity of crude oil $i \in C1$, exported (or sold to another company). $pr_{p,k}$ is the price of product p in the demand sub-region k, and pr_i is the price of crude oil $i \in C1,C2$, at a virtual point '0' on the boundary of the region under analysis. The costs are denoted by: cp_i , crude oil production cost at facility $i \in C1$; ct_{i0} , crude oil transportation cost from facility i to the virtual point 0 ($\forall i \in C1$); ct_{0j} , crude oil transportation cost from the virtual point 0 to refinery j ($\forall i \in C2$); ct_{ij} , crude oil transportation cost from production facility i to refinery j ($\forall i \in C1$); $co^j_{i,om(i)}$, operational cost of refining crude oil i at refinery j, at operating mode $om(i)$; and $ct^p_{j,k}$, transportation cost of product p from refinery j to demand sub-region k. The refining capacity of refinery j is denoted by R_j and the crude oil production capacity of facility i ($\forall i \in C1$) by P_i . d^{pk} is the quantity of product p demanded at sub-region k (at price $pr_{p,k}$). Finally, $r^{j,p}_{i,om(i)}$ denotes the conversion rate from crude oil i to product p at refinery j and operating mode $om(i)$. The profit maximization problem can be now presented:

Maximize

$$\sum_{p=1}^{n_p}\sum_{k=1}^{n_k}\sum_{j=1}^{n_j}\left(\left(pr_{p,k} - ct^p_{jk}\right)\cdot y^{p,j,k}\right) -$$

$$\sum_{j=1}^{n_j}\left(\sum_{i\in C1}\left(\left(cp_i + ct_{ij}\right)\cdot\sum_{om(i)=1}^{n_{om(i)}} x^j_{i,om(i)}\right) + \sum_{i\in C2}\left(\left(pr_i + ct_{0j}\right)\cdot\sum_{om(i)=1}^{n_{om(i)}} x^j_{i,om(i)}\right) + \sum_{i\in C1,C2}\sum_{om(i)=1}^{n_{om(i)}} co^j_{i,om(i)}\cdot x^j_{i,om(i)}\right) +$$

$$\sum_{i\in C1}\left(pr_i - \left(cp_i + ct_{i0}\right)\right)\cdot x^e_i \qquad (3.1)$$

subject to

- refinery capacity constraints:

$$\sum_{i\in C1,C2}\sum_{om(i)=1}^{n_{om(i)}} x^j_{i,om(i)} \le R_j, \qquad \text{for each } j, \qquad (3.2)$$

- crude oil production capacity constraints:

$$x_i^e + \sum_{j=1}^{n_i} \sum_{om(i)=1}^{n_{om(i)}} x_{i,om(i)}^j \le P_i, \qquad \text{for each } i, \tag{3.3}$$

- local demands constraints:

$$\sum_{j=1}^{n_i} y^{p,j,k} \le d^{p,k}, \qquad \text{for each } p, k, \tag{3.4}$$

- conversion rates constraints:

$$z_{i,om(i)}^{p,j} = x_{i,om(i)}^j \cdot r_{i,om(i)}^{j,p}, \qquad \text{for each } j, p, i, om(i), \tag{3.5}$$

- production-shipping conservation constraints:

$$\sum_{k=1}^{n_k} y^{p,j,k} = \sum_{i \in C1,C2} \sum_{om(i)=1}^{n_{om(i)}} z_{i,om(i)}^{p,j}, \qquad \text{for each } j \text{ and } p, \tag{3.6}$$

and

$$x \ge 0, \quad y \ge 0. \tag{3.7}$$

The first term of the objective function is the revenue obtained from selling refined products in all markets, less the transportation costs. The second term is the cost of producing all the refined products expressed as the acquisition costs of crude oil, from self production or bought on the global market, including the transportation of the crude to each refinery, plus the operating costs of each refinery, which depends on the crude oil and the operating mode. The first term less the second gives the profit obtained from producing and selling refined products. The third term is the profit obtained from selling crude oil on the global market, instead of using it in its own refineries. This term must be considered because each crude oil has special characteristics and a particular type may not be adequate for a given refinery, as it has a low value for this refinery, but it may be suitable for other refineries. Then, a firm may realize more profit by selling to the global market a crude oil which is not suitable for its own refineries, but that has a high value for others.

The conversion rates constraints (3.5) establish the quantity of each product produced from each crude oil at each refinery using a given operating mode. The flexibility of a refinery in producing different product mixes from the same crude oil is modeled by different operating modes $om(i)$. For example, three basic operating modes are "maximum kerosene", "maximum gasoline" and "maximum diesel oil". These are commonly used in several kinds of refineries. For each refinery and each crude oil processed, different rates for each product are obtained for each operating mode, giving some flexibility on the refinery operations.

The local demand constraints (3.4) establish a maximum demand for each product p in each sub-region k, associated to the price $pr_{p,k}$. For the export demand sub-region, it is set to a big value for the export demand for all products; then, those constraints became irrelevant to this sub-region. The production-shipping conservation restrictions (3.6) establish flow conservation between production and shipping in each refinery j for each product p. The constraints (3.2) and (3.3) are simple production capacity constraints, commonly used in linear programming.

For the other kinds of firms, the profit-maximization problem can be easily obtained from this one. As an example, for companies that only operate refineries, the set $C1$ is empty and the

third term of the objective function can be deleted. For petrochemical companies, the number of refined products is usually smaller and the crude oil acquisition costs are replaced by the acquisition costs of other inputs. For traders, the acquisitions costs must consider the purchase prices of each product in the global market and transportation costs, and the conversion rates constraints are irrelevant.

3.2 Nonlinear Profit Maximization Formulation

The previous section considered that the price and demand for each product are fixed, and therefore, a linear profit maximization problem could be formulated. This assumption is relaxed by the definition of a demand function dependent on the prices of all products. If this demand function can be inverted, a price function is obtained and, under some symmetry assumptions, a nonlinear profit maximization problem for petroleum companies can be formulated.

In the problem (3.1)-(3.7), the price function for each product in each sub-region (instead of the fixed price) could be considered in the objective function (3.1). The problem becomes nonlinear and the local demand constraints can be deleted since it is now represented in the objective function. This gives the monopolist profit maximization problem. In the oligopoly equilibrium, assuming that each firm has a Cournot-based behavior (Okuguchi & Szidarovsky, 1999), when incorporating the price function into the objective function (3.1), the strategies of other firms (output levels) are considered to be fixed. Then, the price function for a product p in a demand sub-region k is dependent only on the output of a firm f, i.e.,

$$pr_{pk} = pr_{pk}\left(\sum_{j=1}^{n_j} y^{p,j,k} + Q_{F-f}^{pk}\right),$$

where Q_{F-f}^{pk} is the quantity of product p supplied by the other firms ($\neq f$) in the sub-region k. The profit maximization problem of each firm f in the oligopoly game with Cournot-based behavior then takes the form:

Maximize

$$\sum_{p=1}^{n_p}\sum_{k=1}^{n_k}\sum_{j=1}^{n_j}\left(\left(pr_{pk}\left(\sum_{j=1}^{n_j} y^{p,j,k} + Q_{-f}^{pk}\right) - ct_{jk}^{p}\right)\cdot y^{p,j,k}\right) -$$

$$\sum_{j=1}^{n_j}\left(\sum_{i\in C1}\left(\left(cp_i + ct_{ij}\right)\cdot \sum_{om(i)=1}^{n_{om(i)}} x_{i,om(i)}^{j}\right) + \sum_{i\in C2}\left(\left(pr_i + ct_{0j}\right)\cdot \sum_{om(i)=1}^{n_{om(i)}} x_{i,om(i)}^{j}\right) + \sum_{i\in C1,C2}\sum_{om(i)=1}^{n_{om(i)}} co_{i,om(i)}^{j}\cdot x_{i,om(i)}^{j}\right) +$$

$$\sum_{i\in C1}\left(pr_i - \left(cp_i + ct_{i0}\right)\right)\cdot x_i^e \qquad (3.8)$$

subject to (3.2), (3.3), (3.5), (3.6) and (3.7).

This oligopoly game is multi-product, with spatially separated markets, with an implicit cost function formulated as a mathematical program. The manufacturing of the products is not independent, i.e., to produce certain amount of one product, a firm must produce some amount of other products. The capacity constraints consider the total amount of crude oil to be

processed, which corresponds to the total output of refined products. This is different from the assumption made by Okuguchi & Szidarovsky (1999), where the feasible solution set X_f of a firm f is the Cartesian product of the firm's capacity for each product.

By using the problem (3.8) iteratively, a relaxation/diagonalization algorithm can be developed for this "refined products spatial oligopoly game". However, it is not easy to establish *a priori* conditions that guarantee the convergence to the equilibrium.

Conditions on the existence and uniqueness of a solution to (3.8) are easier to analyze, even though rigorous proofs are not given here. Since the price function is expected to be decreasing in the total demand (supplied by all firms) for each product in each sub-region, it will be decreasing as a function of the supply of one firm, considering the other firm's outputs are constant. That is, if the function $pr(x)$ is decreasing in $x \in R^+$, then $pr(x + \text{constant})$ is also decreasing in $x \in R^+$ (Murphy *et al*, 1982). Then, the revenue, for one firm f, $\left(pr_{pk}\left(q_f^{pk} + Q_{F-f}^{pk}\right)q_f^{pk}\right)$, of each product p in each sub-region k is concave, where $q_f^{pk} = \sum_{j=1}^{n_f} y^{p,j,k}$.

The total production and crude oil acquisition cost, for each firm, is increasing in the amount of crude oil. It is expected that the total operating costs of each company are convex. Since refineries with higher costs are used only when the more efficient ones are set to capacity, the refining operating costs are convex. The same occurs with crude oil acquisition costs. For crude oils in $C1$ (own production), first the cheapest production facilities are used until their capacities are met, then, the less efficient ones are used. For crude oils in $C2$, cheaper oils are bought first and the expensive ones next, when the cheaper ones are no longer available. Also, it is expected that crude oils in $C2$ have higher acquisition costs than in $C1$.

Even if the profit function were concave in each component, i.e., in each variable $y^{p,j,k}$, the concavity of the whole problem is not guaranteed. The conversion rates constraints establish that when maximizing the production of a high valued product, such as gasoline, to be sold in a local market at a relatively high price, some amount of other products will be produced. Some of them are low valued, such as fuel oil; and when the local demand is not large enough to consume all the fuel oil available in the local refinery, this product must be exported to other sub-regions. Sometimes, this export operation may yield negative profits, decreasing the total profit of selling all products. Hence, maximizing the production of a high profitable product, from some crude oil in a given operating mode, may not be optimal, when the profitability of the other products is considered.

3.3 More Specialized Petroleum Refining Optimization Programs

The mathematical programs presented in sections 3.1 and 3.2 considered each refinery as a black box, where inputs go in, the operating mode is set and the final products results. However, petroleum companies may use more specialized optimization programs. Some products may be processed by more complex processes than other products, even in the same

refinery. Then, operational costs may differ from one product to another, even if they are produced in the same refinery, with the same crude oil and operating mode.

For companies that operate more than one refinery, it is common to transfer intermediate products from one refinery to another. This is done since the refineries have different equipments, which can process different hydrocarbon chains of crude oil. Usually, the capacity of given equipment is greater than the quantity of intermediate product produced in the refinery. Therefore, intermediate products from refineries that do not have capacity of processing these products can be transferred to other refineries.

Thus, when scheduling its own production, a petroleum company regularly makes use of an optimization program where the operating costs are related to each product, and not only on the crude oil and operating mode used. This optimization program also considers intermediate products, transfers from one equipment to another as well as from one refinery to another. This disaggregation implies that costs must be evaluated for each equipment, with each product processed. The final product operating cost is obtained by aggregating the intermediate costs.

Considering the oligopoly game with petroleum companies, a company that tries to evaluate the strategies of other companies could use a specialized optimization program as its own profit maximization (implicit) function. However, to simulate the profit functions of other companies, insufficient information on costs and equipments is available to use a specialized optimization program. The amount of information needed in a specialized refinery optimization program is often impossible to be estimated by anyone outside the petroleum company. Therefore, for the other companies, simple profit maximization problems such as the ones presented in section 3.1 and 3.2 are used.

3.4 Oligopoly Equilibrium and Policy Intervention

In the spatial price equilibrium presented in Nagurney (1993), the possibility of policy intervention was formulated as a variational inequality problem. This formulation can be extended to the case of oligopoly equilibrium, where the need of policy intervention is expected to be greater. A policy of maximum demand prices (demand price ceiling) may be used and, in this case, an excess demand v_{pk} for product p at demand market k may occur.

The profit function for each firm now has a constraint on the prices pr_{pk} for each product p in each market k:

$$pr_{pk} \leq \overline{pr_{pk}} \text{ , for all } p \text{ and } k, \tag{3.9}$$

where $\overline{pr_{pk}}$ is the demand price ceiling established for each product p and market k.

The profit maximization problems for each petroleum firm must now consider two price functions (for each demand market, and for each product). One function is the regular price function $pr_{pk} = pr_{pk}(.)$, as if no policy intervention is in effect. This function is considered if the total demand in k is greater than $\overline{d_{pk}} = pr_{pk}^{-1}\left(\overline{pr_{pk}}\right)$. In the other way, a constant function $pr_{pk} = \overline{p_{pk}}$ is considered if the total demand in k is smaller than $\overline{d_{pk}} = pr_{pk}^{-1}\left(\overline{pr_{pk}}\right)$. In this case, an excess demand v_{pk} may occur.

4. A Diagonalization Algorithm to the Petroleum Oligopoly Equilibrium

In the literature that was briefly discussed in Section 2, several diagonalization (Jacoby type) algorithms applied to oligopoly equilibration were presented. They were applied to solve the classical oligopoly equilibrium game, where there is only one product and one market, the one-product spatial oligopoly equilibrium game and the multi-product oligopoly equilibrium game. In the case of the multi-product game (Okuguchi & Szidarovsky, 1999), where only one demand market is considered, some convergence results were presented. For this problem, the algorithm based on a sequential adjustment dynamic process has usually better convergence properties, like greater speed of adjustments, than a simultaneous adjustment process (Theorems 4.1.3 and 4.3.2, Okuguchi & Szidarovsky, 1999).

For an oligopoly game with profit maximizing petroleum firms, as presented above, a diagonalization algorithm may be applied to find equilibria. Convergence results are not established *a priori*, given that the problem is a multi-product spatial oligopoly equilibrium. That is, the model is a generalization of the models discussed in Section 2, and the profit functions of each firm have special structure obtained implicitly from a mathematical programming formulation, where the supply of each final product is intimately correlated with the supply of the other products.

This diagonalization algorithm solves a problem such as (3.8) subject to (3.2), (3.3), (3.5), (3.6) and (3.7), for each firm iteratively. Thus, one firm considers the strategies of the others using the results from a previous iteration. However, considering the characteristics of the price function of each product, updates of this function must be done iteratively, to capture cross-price effects when substitute fuels have different changes in prices.

A sequential adjustment process, which is a Gauss-Seidel method, seems to give better convergence results (Okuguchi & Szidarovsky, 1999). It assumes that each firm maximizes its profit by using the latest information available. Let $U_f(t)$ denote the profit function of firm f at time period t, obtained by solving problem (3.8). Assume that there are F firms. Let $\mathbf{q}_f(t)$ denote the output vector of firm f at time period t. This vector is of dimension $(n_p + n_k)$ with components

$$qf^{pk}(t) = \sum_{j=1}^{n_1} y^{p,j,k} . \qquad (4.1)$$

This algorithm is summarized below:

Step 0 (Initialization): Obtain a feasible strategy for each firm, using historical data on prices, demands and supply or even solving (3.8) assuming all other firms' strategies are zero. Set $t = 0$.

Step 1 (Iteration): Solve (3.8) for each firm f assuming

$$Q_{F-f}^{pk} = \sum_{g=1}^{f-1} q_g^{pk}(t) + \sum_{g=f+1}^{F} q_g^{pk}(t-1). \qquad (4.2)$$

After solving (3.8) for each firm, update the price function for each p and k with the new prices for $q \neq p$ in sub region k, capturing cross price effects.

Step 2 (Convergence test): If $(\mathbf{q}_f(t) - \mathbf{q}_f(t\text{-}1)) < \varepsilon$, some arbitrary small tolerance, for all firms, then stop. Otherwise, set $t = t + 1$, and go to step 1.

The price function update, in step 1, could be done in step 2, after all firms had played, and, in this case, firms would be less sensitive to changes in the demand function caused by cross price effects. It can be seen as if the price functions were updated such as a Jacoby method.

5. Application to the Brazilian Oligopolized Petroleum Market

The algorithm presented in the previous section was applied to the Brazilian market of refined products. This application is based on the data available for the month of April 2001. The parameters used in the demand functions consider consumption and prices prevailing during this month, when the Brazilian market was not open to competition, and the Government established the ex-refinery prices for the main products. Price elasticities were estimated based on the time series data from 1977 to 1997. Linear price functions are fitted to data, which include cross price effects.

Three aggregated products are considered: light distillates, medium distillates and heavy distillates[2]. Table 1 shows the price and cross price elasticities for these three products. There are fifteen internal demand sub-regions where the demand functions for these products are evaluated. These sub-regions have been defined to correspond to storage terminals and distribution centers used by distribution companies, where the competition in supplying refined products is expected to occur.

Table 1 – Price elasticities used in the simulation.

Price elasticities	Gasoline	diesel	fuel oil
gasoline	-0.15	0.03	0.00
diesel	0.05	-0.12	0.04
fuel oil	0.00	0.03	-0.40

Thus, a linear price function of the form

[2] The main product in light distillates is gasoline, diesel oil in the medium distillates, and fuel oil in the heavy distillates.

$$pr_{pk}\left(d^{pk}\right) = \frac{pr^0_{pk}}{\varepsilon^{pp}} \cdot \frac{d^{pk}}{d_0^{pk}} + \frac{pr^0_{pk}\left(\varepsilon^{pp} - 1 - \Delta pr_{-pk}\right)}{\varepsilon^{pp}}, \tag{5.1}$$

is obtained for each product p and sub-region k, and is dependent only on the total demand for the product p in the sub-region k, d^{pk}. ε^{pp} is the price elasticity of product p. pr^0_{pk} and d_0^{pk} are known price and demand for product p in sub-region k, in a given period of time. For this application, these prices and demands are based on the data available for the month of April 2001, and are listed in the tables 2 to 7 below. The term Δpr_{-pk} captures cross price effects between different products, using the prices obtained in the previous iteration in the algorithm and the cross price elasticities ε^{pq}. It is defined as:

$$\Delta pr_{-pk} = \sum_{q=1, q\neq p}^{n_p} \varepsilon^{pq} \frac{\Delta pr_{qk}}{pr^0_{qk}}, \text{ where } \Delta pr_{qk} = pr_{qk} - pr^0_{qk}. \tag{5.2}$$

Three refining companies can supply any of the internal demand sub-regions, as well as export to an external sub-region. Transportation costs to any of the sub-regions are evaluated by considering the mode used, and the capacity of each mode and terminal, in the case of multimodal transportation. Two of these companies operate one refinery each, and buy crude oil on the global market. The third company, the larger one, operates 10 refineries and several crude oil production facilities. It supplies more than 90% of the existing refining capacity in Brazil. Figure 1 shows a map of Brazil with the refineries, terminals and the demand sub-regions considered in this example.

Figure 1 – Brazil, demand sub-regions, refineries and terminals.

Seven crude oil types are considered: three produced in Brazil, and four from other countries that usually trade crude oil with Brazil. Acquisition costs for the "foreign" crude oils are estimated based on prices published on the market and on the transportation costs to each refinery. Crude oil production costs were estimated by considering the type of exploitation (on-shore / off-shore).

Each refinery may operate in two modes: "maximum gasoline" and "maximum diesel". Therefore, conversion rates from each crude oil into each refined product were estimated for each refinery by using a given operating mode. This estimation considered the complexity of each refinery and the characteristics of the crude oil, especially density. It was used in estimating refining costs as well.

It is assumed that traders may also supply refined products to any of the 15 internal demand regions, acquiring products basically in two foreign points, Argentina and Venezuela. Two situations are then considered, with one trader and with two traders, in order to show the impact of one extra firm competing in supplying this market. Table 2 gives the actual consumption (d_0^{1k}) for each sub-region in April 2001 for the light distillates, as well as the resulting supplies of the two scenario simulations. Table 3 gives the regulated prices (pr^0_{1k}) for light distillates, practiced in April 2001, and the equilibrium prices in the two scenarios. Tables 4 and 5 give the same results for medium distillates, and Tables 6 and 7 for heavy distillates.

Table 2: Light distillates consumption and supplies [m^3].

		Scenario 1					Scenario 2					
Sub-regions	**D** (Apr/2001)	Ref. 1	Ref. 2	Ref. 3	Trader 1	**Total**	Ref. 1	Ref. 2	Ref. 3	Trader 1	Trader 2	**Total**
PA/AP	28,077	10,384	457	566	9,772	21,178	7,813	335	493	7,201	7,201	23,044
AM/RR/RO/AC	36,631	13,943	279	425	12,746	27,393	10,555	186	399	9,357	9,357	29,853
MA/PI	26,422	9,776	394	497	9,267	19,935	7,339	271	420	6,830	6,830	21,689
CE/RN	53,861	20,040	750	962	18,792	40,544	15,080	549	856	13,832	13,832	44,147
PE/PB/AL	75,001	27,799	1,218	1,543	26,109	56,669	20,929	897	1,353	19,238	19,238	61,655
BA/SE	82,436	31,678	609	941	28,468	61,696	24,071	482	965	20,861	20,861	67,239
RJ/ES	183,231	69,968	1,958	3,538	62,530	137,995	52,952	1,820	3,733	45,805	45,805	150,117
MG	188,306	73,037	1,305	2,942	64,054	141,339	55,537	1,288	3,270	46,855	46,855	153,805
SP W	144,365	56,038	1,363	1,725	49,158	108,284	42,818	1,292	1,918	35,939	35,939	117,907
SP central	200,000	77,108	2,530	2,843	67,967	150,447	58,872	2,346	3,022	49,731	49,730	163,701
SP NE	300,000	114,900	3,949	4,694	102,380	225,923	87,467	3,596	4,885	74,947	74,948	245,843
RS	154,286	58,675	1,902	1,529	54,473	116,579	44,237	1,471	1,378	40,035	40,035	127,156
PR/SC	207,805	79,411	1,443	2,093	73,661	156,607	59,884	866	1,897	54,134	54,134	170,915
MS/MT	47,442	17,102	1,941	2,041	15,194	36,278	13,165	1,650	1,823	11,257	11,257	39,151
GO/DF/TO	118,993	45,104	2,360	2,642	39,726	89,833	34,543	2,078	2,567	29,165	29,165	97,517
Sub-total	1,846,856	704,964	22,458	28,980	634,297	1,390,699	535,260	19,128	28,980	465,185	465,186	1,513,739
External	-	1,415,335	0	0	-	1,415,335	1,586,018	0	0	-	-	1,586,018

Table 3: Light distillates prices [$/m^3].

Sub-regions	Pr (Apr/2001)	Scenario 1	Scenario 2
PA/AP	507.70	1,659.86	1,349.98
AM/RR/RO/AC	492.20	1,634.77	1,331.22
MA/PI	505.50	1,654.04	1,343.13
CE/RN	501.20	1,647.82	1,340.09
PE/PB/AL	507.10	1,650.19	1,340.49
BA/SE	488.00	1,615.80	1,315.58
RJ/ES	489.20	1,600.64	1,302.96
MG	485.30	1,599.21	1,303.70
SP W	486.20	1,602.39	1,305.58
SP central	490.60	1,609.15	1,310.93
SP NE	490.60	1,605.83	1,306.76
RS	487.80	1,587.88	1,283.56
PR/SC	485.60	1,588.26	1,284.06
MS/MT	576.20	1,818.92	1,500.13
GO/DF/TO	512.60	1,665.90	1,362.59

The total supplies in the internal sub-regions increase from Scenario 1 to Scenario 2, but they are considerably small compared to the actual demand with regulated prices. The prices drop from Scenario 1 to Scenario 2 as a result from the higher level of competition. However, they are still much higher than the regulated prices practiced in April 2001. It can also be seen that the refiners supply relatively more quantity of products in the sub-regions closed to their refineries, and the traders are more competitive in the sub-regions where there are no refinery and are more accessible to imported products.

Table 4: Medium distillates consumption and supplies [m^3].

		Scenario 1					Scenario 2					
Sub-regions	**D** (Apr/2001)	Ref. 1	Ref. 2	Ref. 3	Trader 1	**Total**	Ref. 1	Ref. 2	Ref. 3	Trader 1	Trader 2	**Total**
PA/AP	87,598	34,379	678	346	32,252	67,655	25,581	350	241	23,453	23,453	73,079
AM/RR/RO/AC	99,075	39,818	0	0	36,392	76,210	29,767	0	0	26,341	26,341	82,449
MA/PI	68,072	26,759	55	0	25,540	52,354	19,717	0	0	18,498	18,498	56,714
CE/RN	67,872	26,908	308	47	24,898	52,161	20,065	151	65	18,055	18,055	56,390
PE/PB/AL	101,936	39,941	847	508	37,624	78,920	29,696	425	344	27,379	27,379	85,223
BA/SE	190,938	77,272	655	0	69,366	147,293	58,147	361	118	50,241	50,241	159,108
RJ/ES	227,833	91,017	976	1,127	82,658	175,777	68,443	660	1,404	59,765	59,765	190,036
MG	262,329	106,476	190	365	94,903	201,934	80,471	22	885	68,525	68,525	218,427
SP W	283,730	115,718	783	0	102,406	218,907	87,444	727	0	74,131	74,131	236,433
SP central	205,841	83,591	946	0	74,268	158,804	63,082	850	88	53,759	53,759	171,540
SP NE	306,951	123,973	1,661	5	111,154	236,794	93,227	1,356	497	80,408	80,408	255,896
RS	240,805	97,064	1,296	0	86,988	185,349	72,957	1,260	0	62,881	62,881	199,980
PR/SC	402,002	162,126	1,374	0	145,892	309,392	121,747	1,004	121	105,512	105,512	333,897
MS/MT	211,163	79,489	9,209	8,270	71,574	168,542	60,544	7,520	7,033	52,630	52,630	180,357
GO/DF/TO	171,335	68,132	2,883	2,011	60,636	133,662	51,565	2,335	1,884	44,068	44,068	143,920
Sub-total	2,927,482	1,172,662	21,862	12,679	1,056,551	2,263,754	882,451	17,022	12,679	765,648	765,648	2,443,448
External	-	1,008,684	0	0	-	1,008,684	1,335,228	0	0	-	-	1,335,228

Table 5: Medium distillates prices [$/m^3].

Sub-regions	Pr (Apr/2001)	Scenario 1	Scenario 2
PA/AP	426.30	1,771.84	1,415.02
AM/RR/RO/AC	415.80	1,746.66	1,395.16
MA/PI	418.10	1,746.14	1,385.73
CE/RN	418.50	1,756.53	1,404.91
PE/PB/AL	427.80	1,764.02	1,405.73
BA/SE	418.00	1,742.66	1,393.76
RJ/ES	416.00	1,718.25	1,369.92
MG	412.60	1,717.44	1,371.70
SP W	414.50	1,719.24	1,375.03
SP central	415.70	1,721.43	1,376.29
SP NE	415.70	1,719.01	1,372.02
RS	412.40	1,703.01	1,358.96
PR/SC	414.20	1,715.21	1,368.50
MS/MT	504.50	1,987.56	1,610.39
GO/DF/TO	440.90	1,799.20	1,443.92

Similar results to the light distillates are obtained to the medium distillates. The total supplies increase and the prices drop from Scenario 1 to Scenario 2.

Table 6: Heavy distillates consumption and supplies [m^3].

		Scenario 1					Scenario 2					
Sub-regions	**D** (Apr/2001)	Ref. 1	Ref. 2	Ref. 3	Trader 1	**Total**	Ref. 1	Ref. 2	Ref. 3	Trader 1	Trader 2	**Total**
PA/AP	49,458	20,623	0	232	14,680	35,535	16,662	155	0	10,719	10,719	38,254
AM/RR/RO/AC	31,627	14,042	0	0	8,847	22,889	11,630	0	0	6,435	6,435	24,500
MA/PI	8,840	3,613	0	0	2,829	6,443	2,849	0	0	2,065	2,065	6,978
CE/RN	1,983	840	5	22	557	1,424	685	23	16	402	402	1,529
PE/PB/AL	7,650	3,207	0	53	2,329	5,588	2,584	15	5	1,707	1,707	6,018
BA/SE	19,706	9,085	0	157	5,215	14,456	7,603	222	154	3,733	3,733	15,445
RJ/ES	82,780	37,210	758	3,212	21,585	62,765	31,024	1,810	3,241	15,398	15,398	66,872
MG	111,309	51,248	373	3,672	28,165	83,458	42,470	2,051	3,975	20,110	20,110	88,715
SP W	89,033	42,188	883	1,061	22,615	66,747	35,638	2,117	1,196	16,064	16,064	71,080
SP central	80,000	37,420	1,189	1,018	20,379	60,006	31,524	2,288	1,129	14,483	14,483	63,909
SP NE	100,000	45,676	1,768	1,969	25,986	75,398	38,219	3,055	2,020	18,529	18,529	80,351
RS	31,697	14,085	734	97	9,460	24,377	11,500	920	0	6,875	6,875	26,169
PR/SC	52,335	23,533	189	438	15,828	39,988	19,259	490	93	11,554	11,554	42,949
MS/MT	5,805	2,529	0	0	1,287	3,816	2,174	0	0	933	933	4,040
GO/DF/TO	37,119	17,291	70	144	9,195	26,700	14,603	628	244	6,507	6,507	28,490
Sub-total	709,342	322,590	5,967	12,075	188,958	529,590	268,422	13,774	12,075	135,514	135,514	565,299
External	-	1,167,118	0	0	-	1,167,118	1,163,796	0	0	-	-	1,163,796

Table 7: Heavy distillates prices [$/m^3].

Sub-regions	Pr (Apr/2001)	Scenario 1	Scenario 2
PA/AP	304.00	589.91	529.04
AM/RR/RO/AC	304.00	586.94	528.97
MA/PI	304.00	582.57	516.85
CE/RN	304.00	591.19	531.82
PE/PB/AL	304.00	580.02	518.20
BA/SE	304.00	578.73	521.58
RJ/ES	304.00	559.13	502.33
MG	304.00	566.27	511.27
SP W	304.00	566.00	510.09
SP central	304.00	565.56	509.55
SP NE	304.00	562.45	505.78
RS	304.00	550.88	488.88
PR/SC	304.00	554.91	492.83
MS/MT	304.00	631.46	585.07
GO/DF/TO	304.00	587.57	532.54

As expected, the total supplies for heavy distillates increase from Scenario 1 to Scenario 2, and the prices drop.

The numerical results obtained are for illustration purposes only. The assumptions made on transportation and production costs, refining operating modes, and conversion rates are simplified and their values do not correspond to actual values. However, they are of the right order and magnitude. Also, some taxes, which are levied on the sales of certain petroleum products, are not considered in the analysis. The numerical example demonstrates the tractability of large problems and the expected decrease in prices when a second trader takes part in the game and the market becomes more competitive.

6. Conclusions

A model to analyze the multi-product spatial price oligopoly problem where the supply functions are not explicitly defined was presented. In this model, the supply functions may be defined implicitly from mathematical programming. This model was applied to an oligopoly problem for the petroleum refined products market. Some kinds of firms in this market have complex profit maximization problems, where the production of one product is strictly related to the production of others, given the production characteristics of the crude oil refining process.

An algorithm was presented as well as its economic interpretation as a dynamic sequential adjustment process. This algorithm was applied to the Brazilian petroleum refined products market, which is been opened to competition and is characterized by a small number of firms supplying the market. It was considered that the original refiners will stay in the market and two scenarios were presented: one with only one trader firm competing to supply the market with imported products, and other with two traders. Equilibrium supplies and prices were presented, and the decrease in the equilibrium prices when the second trader takes part in the game can be evaluated, as a result of increased competition.

The possible competition of new entrants in the market, pressuring prices down, is an interesting extension of this model, where the original firms in the game should not set prices based only in the actual players. This is the subject of a forthcoming paper.

Acknowledgments

A suggestion made by Professor Sílvio Hamacher, from Pontifícia Universidade Católica do Rio de Janeiro, Brazil, has contributed to simplify the mathematical problems presented in sections 3.1 and 3.2, and to reduce considerably the computational solution time of the problem.

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